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Energy error based numerical algorithms for Cauchy problems for nonlinear elliptic or time dependent operators

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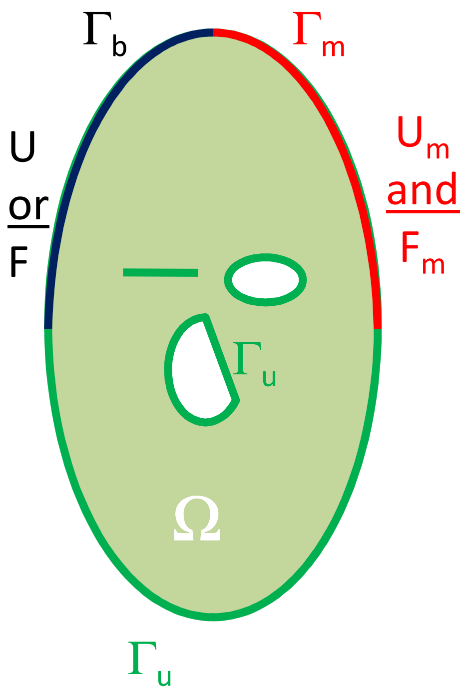
Overview

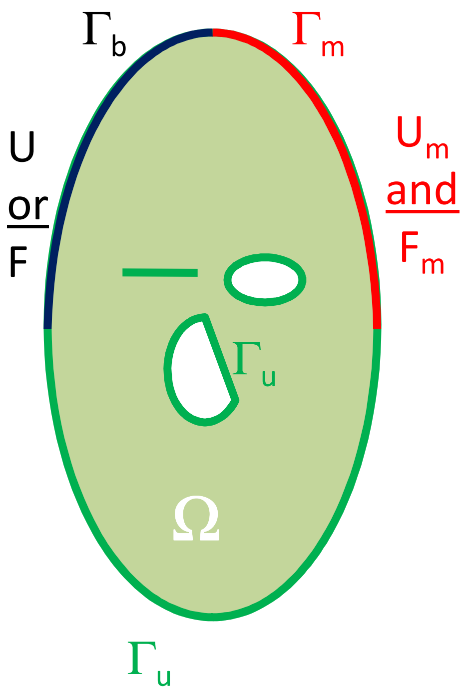
1. A variant of the Cauchy Problem and its applications
2. Solution of the CP by splitting of fields and energy-like error minimization
3. The pseudo-energy for various kind of operators
4. Numerical examples

The Cauchy problem … and its applications



- A variant of the Cauchy problem :
  - Basic ingredients :
    - A Domain Ω of IR^n and eventually a time interval $[0,D]$
    - A PDO A acting on a field u defined on Ω x $]0,D[$ or Ω
    - A boundary operator B associated with A
    - A part Γ_m of the boundary of Ω ($\partial\Omega = \Gamma_m \cup \Gamma_u \cup \Gamma_b$)
    - A field f or U given on Γ_b x $]0,D[$ or Γ_b
    - A pair of fields given on Γ_m : (U_m, F_m)
  - The Cauchy problem : Find u in Ω x $]0,D[$ or Ω s.t. :
    - $Au=0$ in Ω x $]0,D[$ or Ω
    - $Bu=f$ or $u=U$ on Γ_b x $]0,D[$ or Γ_b (*usual BC*)
    - $u= U_m$ and $Bu= F_m$ on Γ_m x $]0,D[$ or G_m (*overspecified BC*)

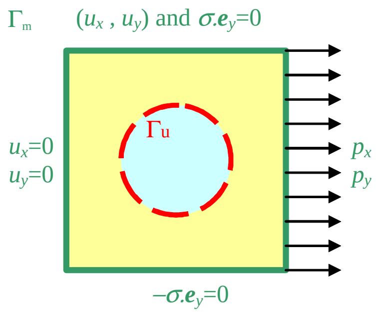
The Cauchy problem … and its applications



- The Γ_u part of the boundary is not accessible, and no BC is known. The Cauchy problem is then viewed
  - as a field extension problem into the domain : determine the whole field *u* inside the body from the knowledge of overspecified data on a part of its boundary.
  - as a data completion problem or a BC identification problem: find *Bu* and *u* on Γ_u from the knowledge of overspecified
- The second version shows that the Cauchy problem for bounded domains pertains to the field of inverse or identification problems.
- The Cauchy problem is the way of setting well-posed time evolution problems (hyperbolic or parabolic ones with Γ_u=Ω) but is ill-posed for others operators and/or others varieties Γ_u

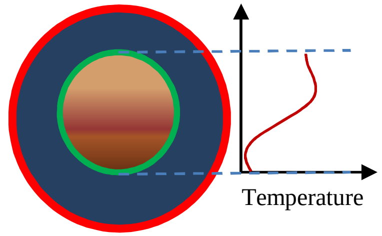
Applications : Elliptic operators

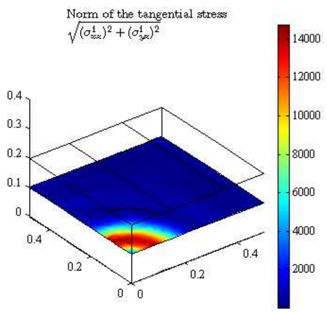
Thermal conduction, Elasticity , Stokes or Darcy system



Determination of elastic parameters for an inclusion with known position and geometry

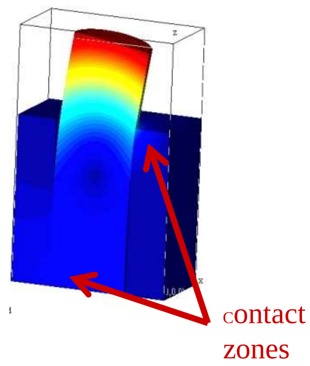
Identification of fluid stratification within a pipe

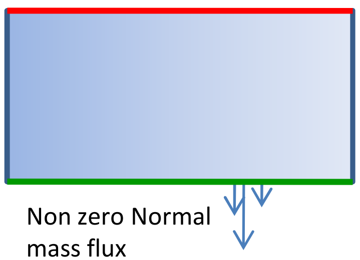




Evaluation of interface stresses

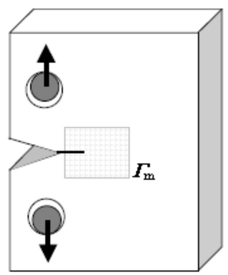
Identification of contact zone and pressure





Determination of leakage in a Darcy system

Determination of linear fracture mechanics parameters from external surface measurements

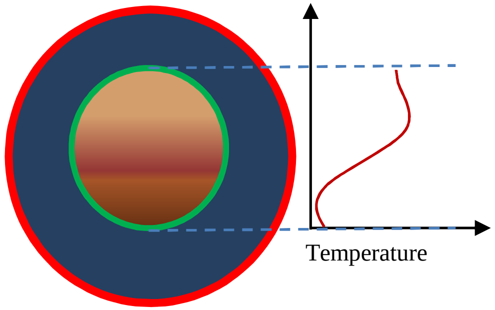


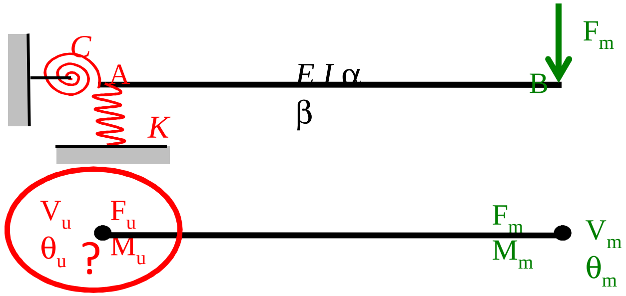
Note that the relation between the boundary terms on Γ_u can be highly non linear

Applications : time dependant, non linear operators

Heat equation, Elastodynamics, Stokes system, hyperelasticity

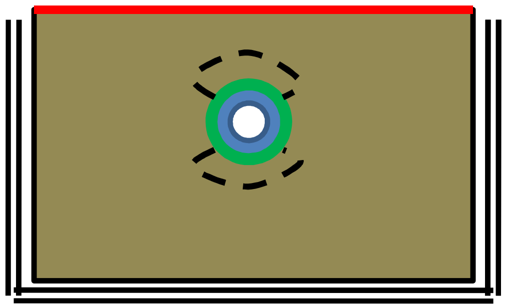
Identification of fluid stratification evolution within a pipe

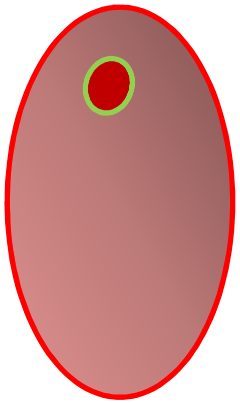




Identification of varying BC for a structure

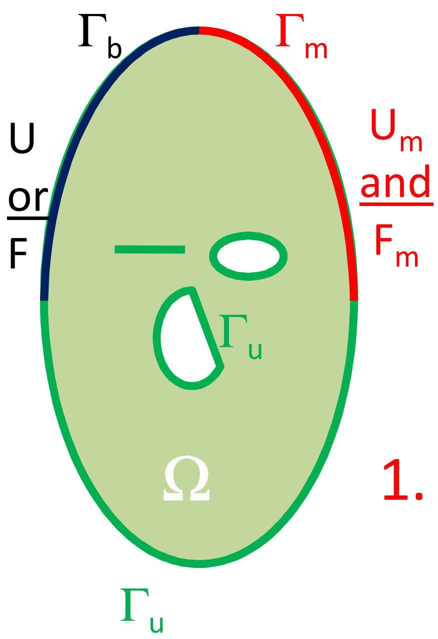
Determination of traction zones in a non linear elastic medium around a tunnel and identification contact zone at the interfaces





Determination physical characteristics of an inclusion with known shape and position for non linear incompressible media

Solution of the CP by splitting the fields and a (pseudo) energy error minimization



Two simple ideas

Model Problem :
the Laplace operator

$$
\begin{cases}
\Delta u = 0 & in\ \Omega \\
\nabla u.n = 0 & on\ \Gamma_b \\
u = U_m, \nabla u.n = F_m & on\ \Gamma_m \\
u = \tau, \nabla u.n = \eta & on\ \Gamma_u
\end{cases}
$$

1. Introduce the BC on Γ_u as unknowns and define two solutions of well-posed problems, using one of the overspecified BC on Γ_m and one of the lacking BC on Γ_u

$$
\begin{cases}
\Delta u_1 = 0 & in\ \Omega \\
\nabla u_1.n = 0 & on\ \Gamma_b \\
u_1 = U_m & on\ \Gamma_m \\
\nabla u_1.n = \eta & on\ \Gamma_u
\end{cases}
\qquad
\begin{cases}
\Delta u_2 = 0 & in\ \Omega \\
\nabla u_2.n = 0 & on\ \Gamma_b \\
\nabla u_2.n = F_m & on\ \Gamma_m \\
u_2 = \tau & on\ \Gamma_u
\end{cases}
$$

Unknows : η and τ

Solution of the CP by splitting the fields and a (pseudo) energy error minimization

If u_1 and u_2 are equal
then the CP is solved : $u = u_1$ and the BC on Γ_u are (η,τ)

2. Introduce a (semi) norm on $(u_1 - u_2)$ and minimize it.

$$E(\eta,\tau) = \frac{1}{2}\int_\Omega \left(\nabla u_1(\eta) - \nabla u_2(\tau)\right)^2 \Longrightarrow \left(u\big|_{\Gamma_u}, \nabla u.n\big|_{\Gamma_u}\right) = \underset{\eta,\tau}{ArgMin}\, E(\eta,\tau)$$

The Energy error *E* is closely related to the physics of the problem and is preferred to the usual least-square error (take into account the eventual heterogeneity, anisotropy …..

Some nice properties of the (pseudo) energy error

- The Error energy is quadratic convex positive
- The minimum is zero (for compatible pair of data (U_m, F_m))
- *E* has an expression involving only boundary terms (used for computation):

$$E(\eta,\tau)=\frac{1}{2}\int_{\partial\Omega}(\nabla u_1(\eta)-\nabla u_2(\tau)).n(u_1(\eta)-u_2(\tau))$$

$$E(\eta,\tau)=\frac{1}{2}\int_{\Gamma_m}(\nabla u_1.n-F_m)(U_m-u_2)+\frac{1}{2}\int_{\Gamma_u}(\eta-\nabla u_2.n)(u_1-\tau)$$

- The term energy is (abusively here) coined from the variational properties associated with the Laplace operator:

$$u=\underset{V}{ArgMin}\ \frac{1}{2}a(v,v)-l(v) \qquad a(u,v)=\int_{\Omega}\nabla u.\nabla v$$

Practical implementation of the energy error method

- Use iterative minimization methods rather than first-order optimality conditions (as soon as the number of unknowns increases) : CG with trust region methods.
- Compute the gradient by (2) adjoint problems : Laplace problems with boundary terms only. Each iteration needs 4 resolutions of direct problems of same kind.
- The fixed-point two-steps algorithm of Kozlov-Maz'ya-Fomin (1991) can be interpreted as an alternating directions descent method for the minimization of *E (so that far better performances are achieved with a "serious" minimizing algorithm)*

The pseudo-energy for various kind of operators : elliptic operators

- For linear elliptic symmetric operators, the pseudo energy as the same form as for the Laplace operator and uses the associated symmetric bilinear form : $E(\eta,\tau) = a\left[u_1(\eta) - u_2(\tau), u_1(\eta) - u_2(\tau)\right]$

- For non-linear elliptic operator associated with a convex potential (like hyperelasticity)

$$\begin{cases} div\,\sigma = 0 & in\ \Omega \\ \sigma \in \partial\varphi(\varepsilon), \varepsilon = \nabla u^s & in\ \Omega \\ \sigma(u).n = F_m,\ u = U_m & on\ \Gamma_m \end{cases}$$

- use the Fenchel cross-residual

$$E(\eta,\tau) = \int_\Omega \left(\sigma_1 - \sigma_2, \varepsilon(u_1) - \varepsilon(u_2)\right) d\Omega$$

Reduces to the error energy for quadratic φ

$$E(\eta,\tau) \geq 0$$

$$E(\eta,\tau) = 0 \Leftrightarrow \quad \sigma_1 \in \partial\varphi\left[\varepsilon(u_2)\right],\ \sigma_2 \in \partial\varphi\left[\varepsilon(u_1)\right]$$

$$Then\ \varepsilon(u_2) = \varepsilon(u_1)\ if\ \varphi\, is\, differentiable$$

The pseudo-energy for various kind of operators : parabolic operators

- The model problem is the heat equation

$$\begin{cases} \rho c\dot{u} - div(k\nabla u) = 0 & in \ \ \Omega \text{x} \,]0,D[\\ u = U_m, k\dfrac{\partial u}{\partial n} = \Phi_m & on \ \Gamma_m \text{x} \,]0,D[\\ u(x,0) = u_0(x) & in \ \ \Omega \end{cases}$$

- The energy error is integrated over the time interval and a control on the final state (at *t=D*) is introduced

$$E(\eta,\tau) = \int_0^D \int_\Omega k\left[\nabla\left(u_1 - u_2\right)\right]^2 d\Omega dt + \frac{1}{2}\int_\Omega \rho c\left[u_1 - u_2\right]_{t=D}^2 d\Omega$$

- The energy error enjoys again a boundary expression

$$E(\tau,\eta) = \int_0^D \int_{\Gamma_u} (u_1 - \tau)(\eta - k\nabla u_2.n) d\Gamma_u dt + \int_0^D \int_{\Gamma_m} (U_m - u_2)(k\nabla u_1.n - F_m) d\Gamma_m dt$$

The pseudo-energy for various kind of operators : hyperbolic operators

- The model problem is the elastodynamics equation
$$\begin{cases} \rho\ddot{\boldsymbol{u}} + c\dot{\boldsymbol{u}} - div(\mathbf{A}:\varepsilon(\boldsymbol{u}) + \mathbf{B}:\varepsilon(\dot{\boldsymbol{u}})) = 0 & in\ \ \Omega \mathrm{x}\,]0,\mathrm{D}[\\ [\mathbf{A}:\varepsilon(\boldsymbol{u}) + \mathbf{B}:\varepsilon(\dot{\boldsymbol{u}})].\boldsymbol{n} = F_m,\ \boldsymbol{u} = U_m & on\ \Gamma_m\,\mathrm{x}\,]0,\mathrm{D}[\end{cases}$$
$$\boldsymbol{u}(x,0) = \boldsymbol{u^0}(x),\ \dot{\boldsymbol{u}}(x,0) = \boldsymbol{u^1}(x)\ in\ \Omega$$

- The energy error contains a control over the time interval of the dissipation, and a control on the final state (at *t=D*) for the elastic and kinetic energies :

$$J(v) = \int_0^D \int_\Omega \left(c\dot{v}^2 + \mathrm{B}:\varepsilon(\dot{v}):\varepsilon(\dot{v})\right) d\Omega dt + \frac{1}{2}\int_\Omega \left(\rho\dot{v}^2 + \mathrm{A}:\varepsilon(v):\varepsilon(v)\right) d\Omega \Big|_{t=D}$$
$$E(\eta,\tau) = J\left[u_1(\eta) - u_2(\tau)\right]$$

- The energy error enjoys again a boundary expression

$$J(v) = \int_0^D \int_{\Gamma_m} (\mathrm{A}:\varepsilon(v) + \mathrm{B}:\varepsilon(\dot{v})).\dot{v}.n\, d\Gamma dt + \int_0^D \int_{\Gamma_u} (\mathrm{A}:\varepsilon(v) + \mathrm{B}:\varepsilon(\dot{v})).\dot{v}.n\, d\Gamma dt$$

A remark for non dissipative media

- For purely elastic media (B=c=0) , the previous energy error degenerate into :

$$J(v) = \frac{1}{2}\int_\Omega \left(\rho \dot{v}^2 + \mathrm{A} : \varepsilon(v) : \varepsilon(v)\right) d\Omega \Big|_{t=D}$$

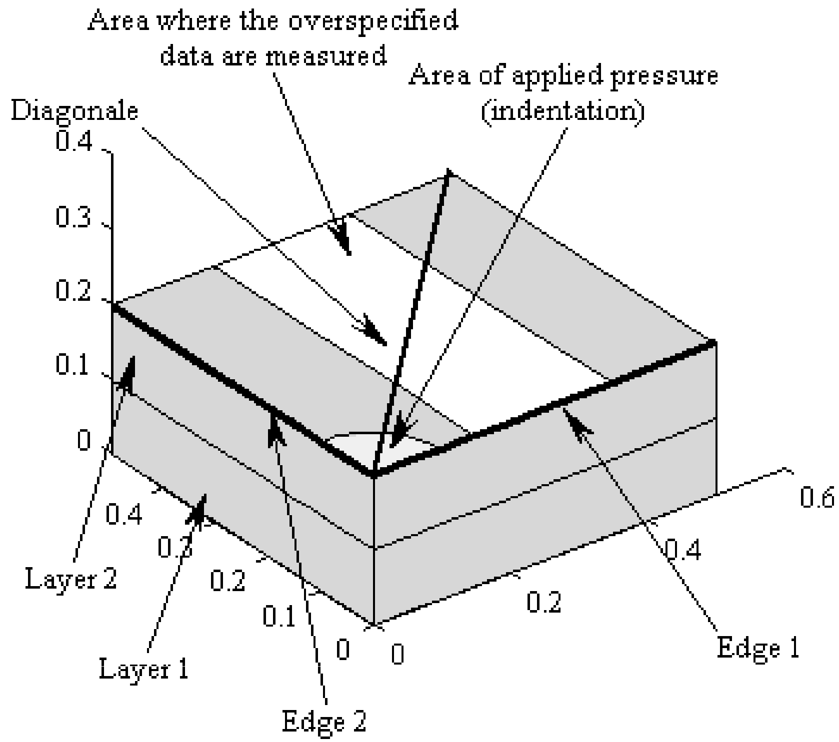
- But due to the exact controllability of the elastodynamics equation (Lions, 1988) this error is no more sufficient :

$$J(v) = 0 \not\Rightarrow v = 0$$

- And a penalized energy must be employed (k,K pseudo material parameters) :

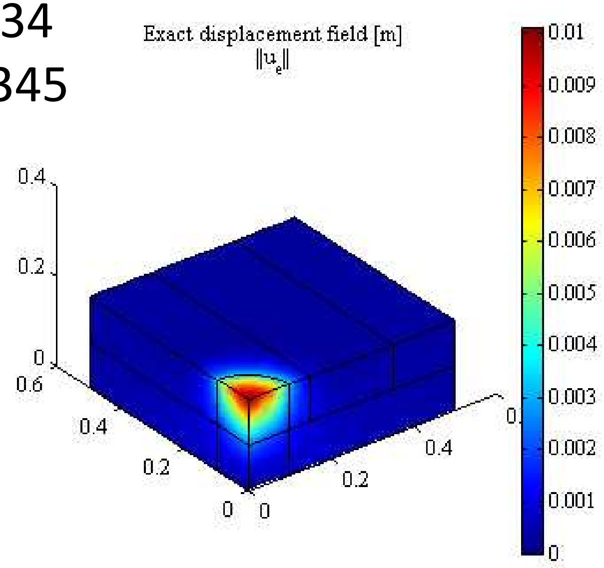
$$J(v) = \frac{1}{2}\int_\Omega \left(\rho \dot{v}^2 + \mathrm{A} : \varepsilon(v) : \varepsilon(v)\right) d\Omega \Big|_{t=D} + \frac{\alpha}{2}\int_0^D \int_{\partial\Omega} \left(k v^2 + K \dot{v}^2\right) d\Gamma dt$$

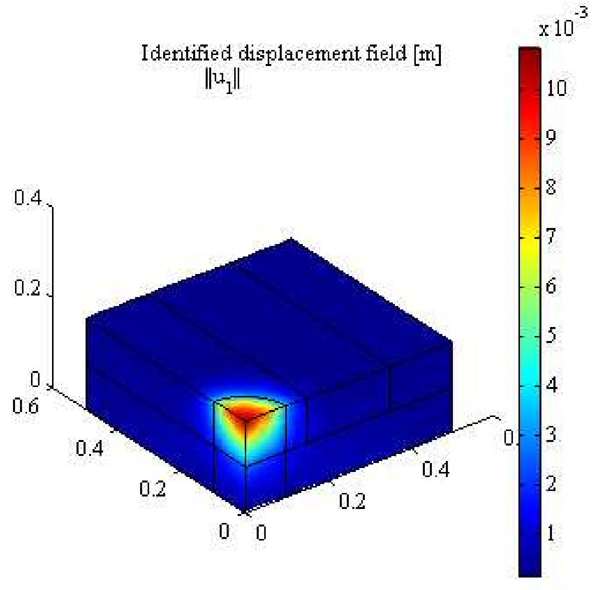
Numerical examples



Indentation of an elastic bi-material :

- No information about the indentation force or indentor displacement
- "displacement field" measurements on a small part of the stress free surface
- Number of node on Γ_m : 134
- Number of nodes on Γ_u : 345
- 185 iterations

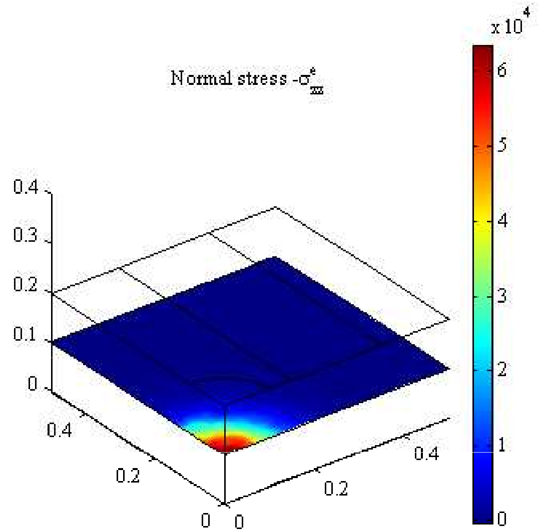


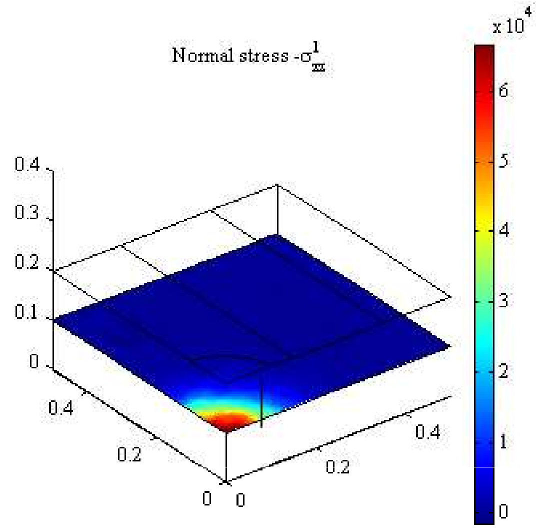


Comparison of exact and identified displacement field

Numerical Examples

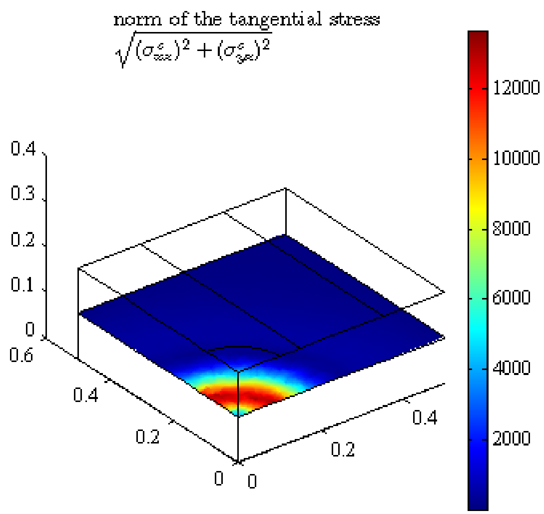
Indentation of an elastic bi-material

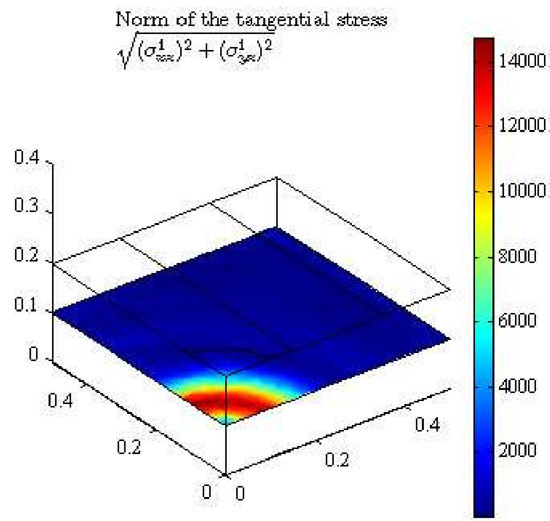




Comparison between identified and computed stress components on the interface

Normal stresses





Norms of shear stresses

Numerical Examples

Identification of inner temperature and heat flux in a pipe : heat equation

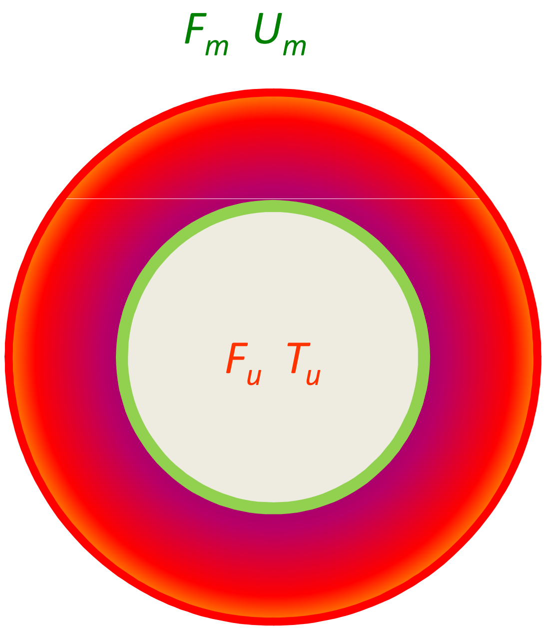
Two-dimensional model for pipeline :

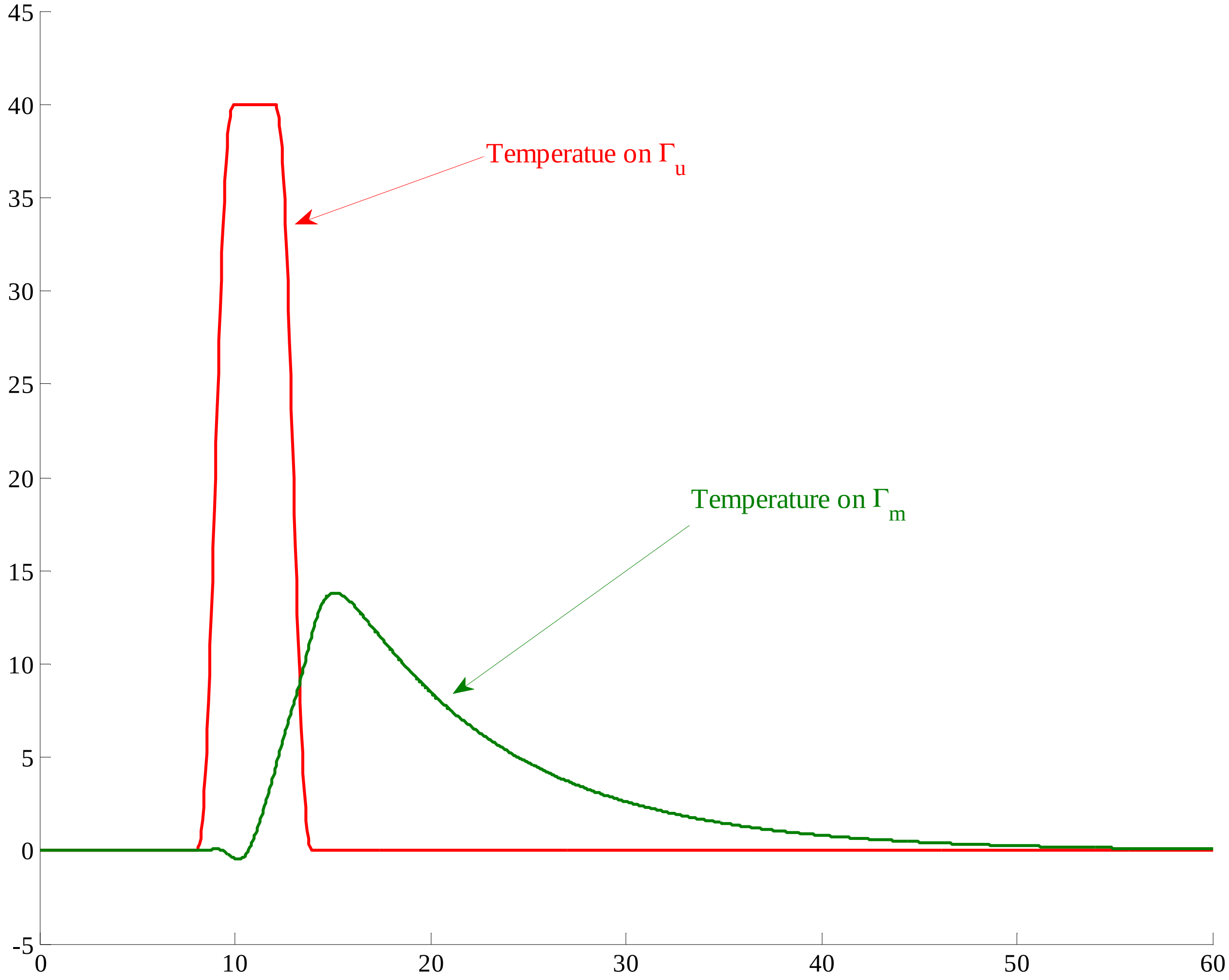
Inner radius R_i=12 cm.

Outer radius R_e = 12.92 cm.

Material properties:

k = 15.9×10^{-2}, ρ = 7.8×10^{-3} Kg/cm^3, c = 494

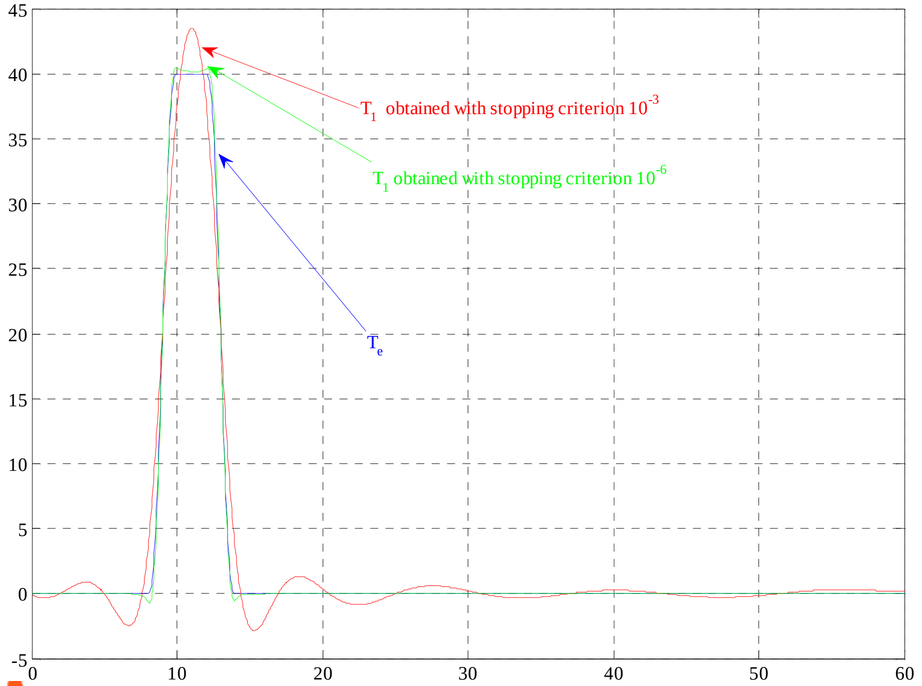




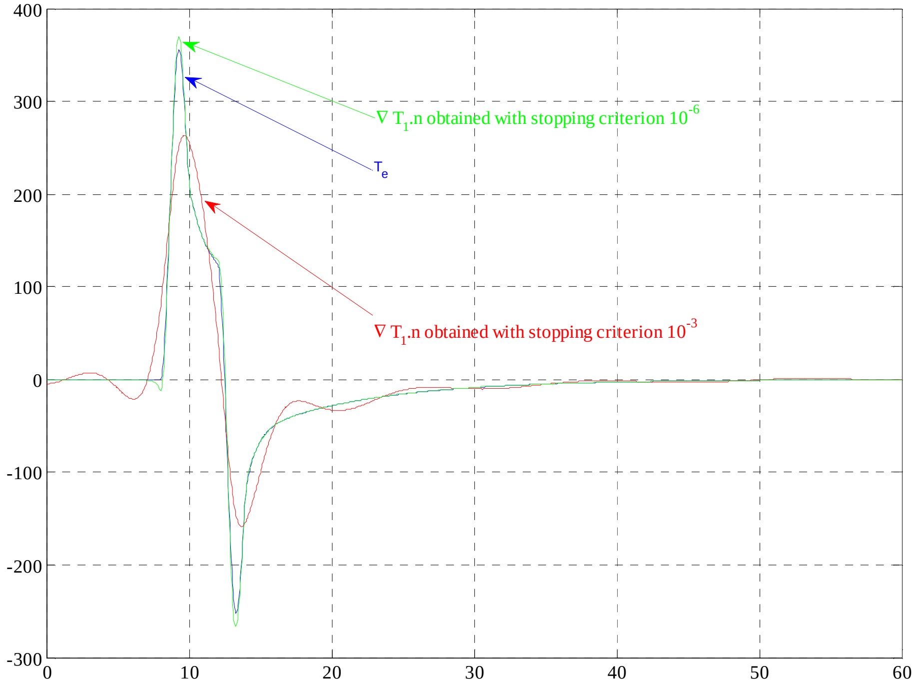
Numerical Examples

Identification of inner temperature and heat flux in a pipe : heat equation

Influence of the stopping criterion



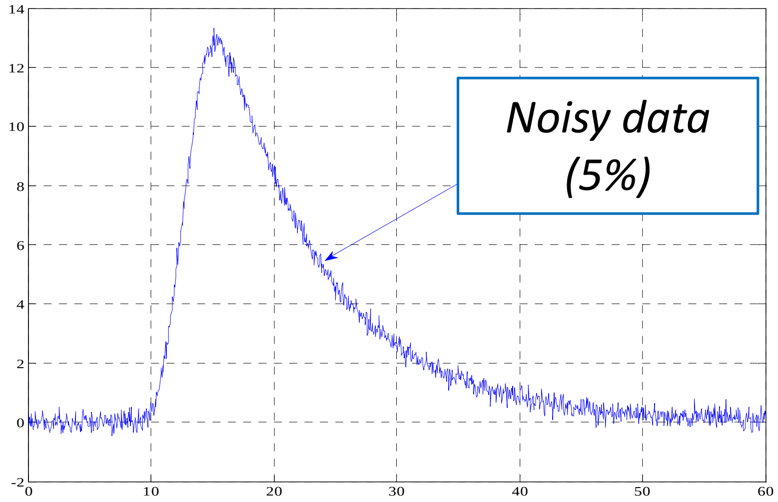
Comparison between identified and computed inner temperature

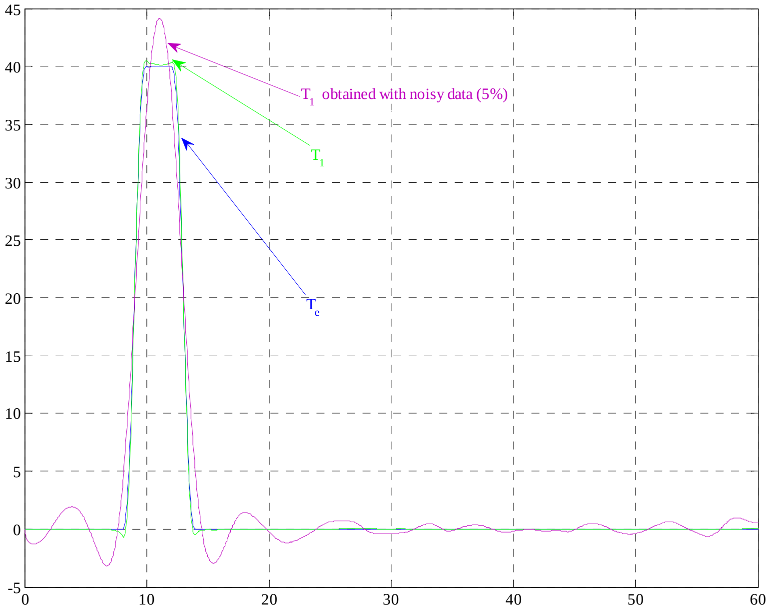


Comparison between identified and computed inner heat flux

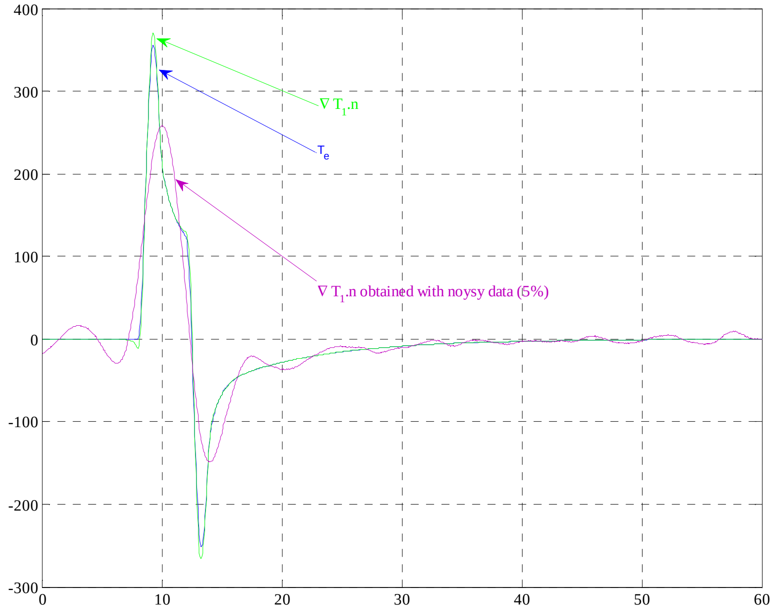
Numerical Examples

Identification of inner temperature and heat flux in a pipe : heat equation





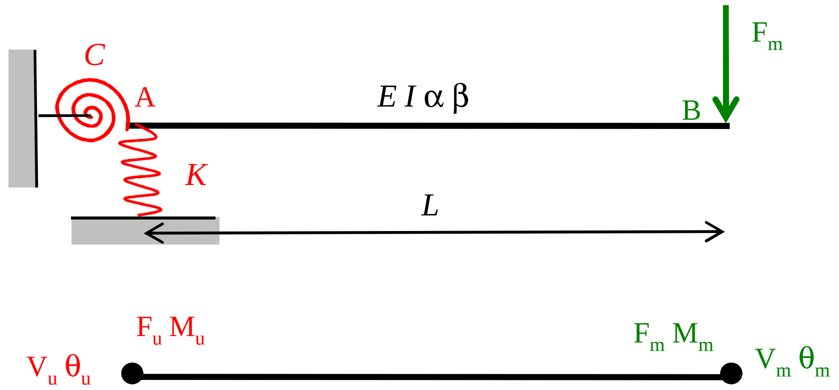
Comparison between identified and computed inner temperature

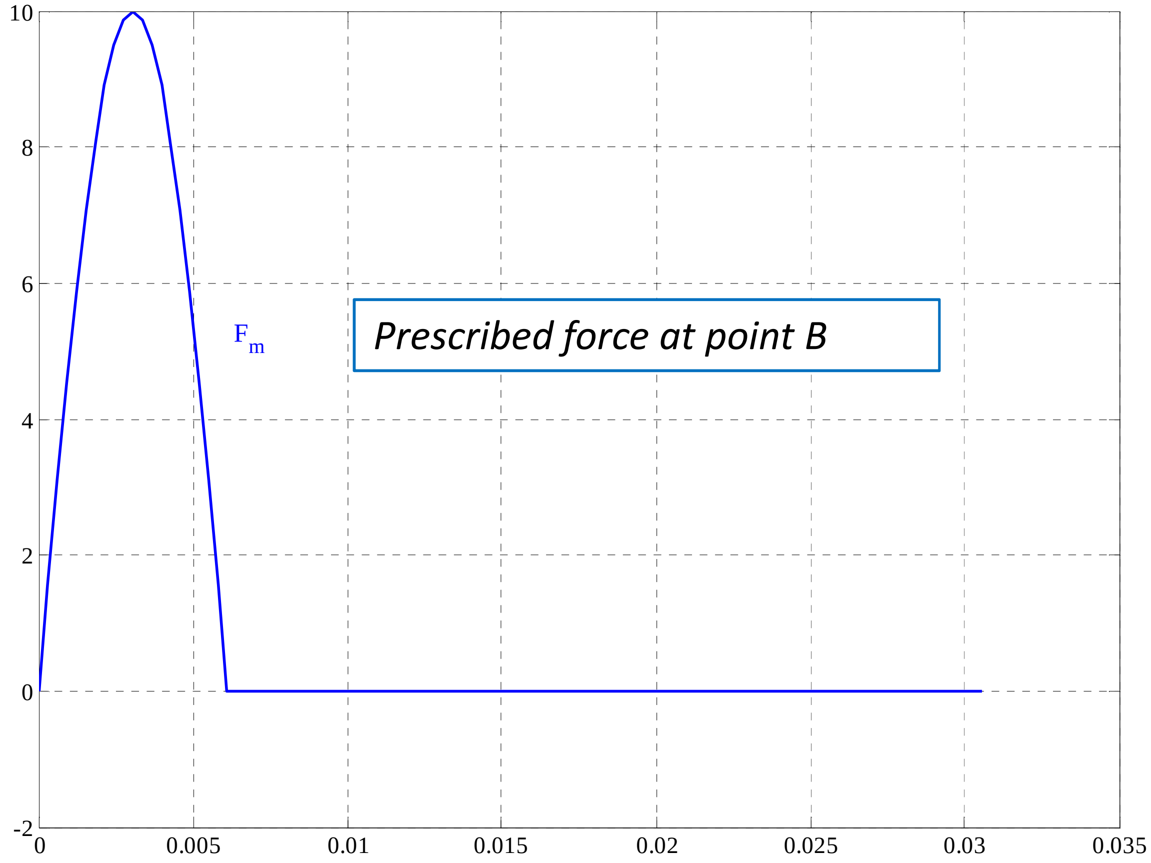


Comparison between identified and computed inner heat flux

Numerical Examples

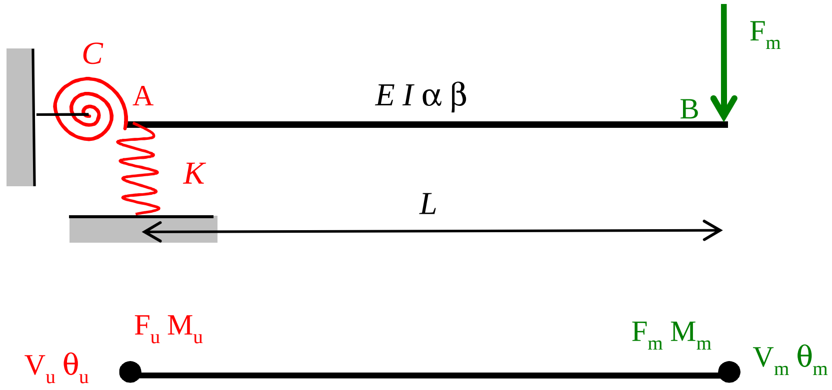
Identification of BC for a beam under an impact loading : elastodynamics

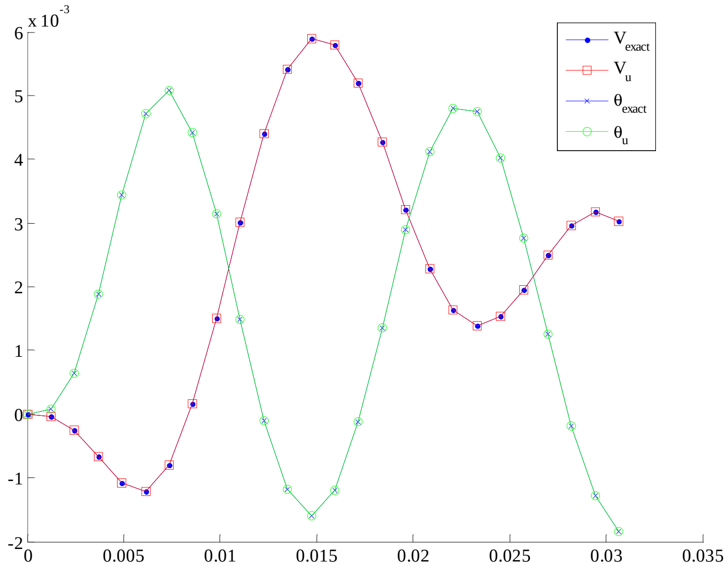




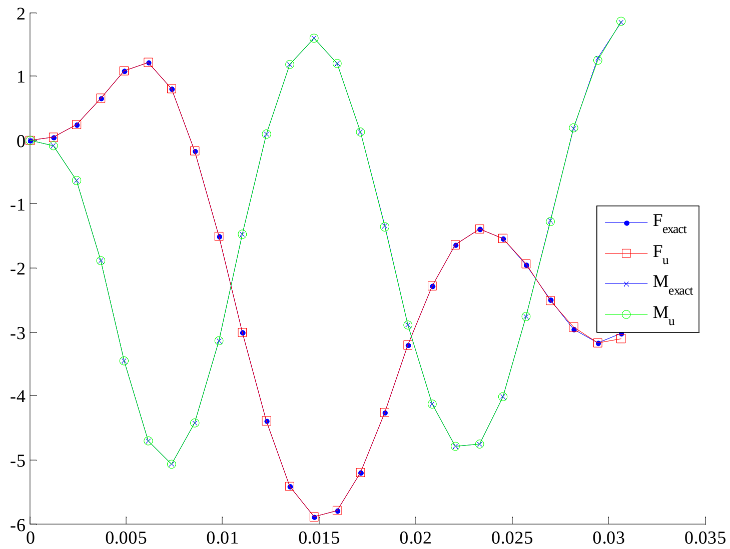
Numerical Examples

Identification of BC for a beam under an impact loading : elastodynamics





Comparison between identified and computed displacement and rotation at point A



Comparison between identified and computed moment and shear resultant at point A

Conclusions

- The method is quite general for (spatial) symmetric operators and lead to efficient functionals compared to least-squares based ones.
- It allows for dealing with 3D applications, and can be used with usual FEM softwares allowing monitoring procedures.
- Regularization is *a priori* needed (the CP is severely ill-posed) but with moderate noise it can be avoided because of smoothing behavior of the minimizing procedure and good behavior of the energy error functional
- Future work on computational efficiency, extension to non-symmetric operators, others applications and resolution of some troubles with adjoint techniques for time dependent and non-differentiable problems (non C^2 convex potentials).